On new explicit Riemannian SU(2(n+1))-holonomy metrics. [1]

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Abstract

We construct in an explicit algebraic form a family of complete noncompact Ricci-flat metrics which generalize Calabi metrics in real dimension $4(n+1)$ and with holonomy $SU(2(n+1))$.

Key words: special holonomy, Calabi metrics.

1 Introduction.

This article is concerned with an exploration of the Ricci-flat Riemannian metrics with exceptional holonomies and naturally continues a number of works [1, 2, 3, 4]. In [4] we were studying Riemannian metrics with $Spin(7)$-holonomy on the cones over 3-Sasakian 7-manifolds and were able to find in an explicit form a continuous family of complete noncompact 8-dimensional metrics $\bar{g}_\alpha$ depending on real parameter $0 \leq \alpha \leq 1$. Metric $\bar{g}_0$ coincides with Calabi $SU(4)$-holonomy metric; metric $\bar{g}_1$ coincides with hyperkähler Calabi metric with holonomy $Sp(2) \subset SU(4)$. Every found metric $\overline{g}_\alpha,\ 0 < \alpha < 1$ is $SU(4)$-holonomy metric and automatically Ricci-flat. Thus, Calabi metrics are "connected" by the obtained one-dimensional family.

However Calabi metrics (firstly appeared in [5]) are both correctly defined not only for dimension 8 but for any dimension divisible by 4. And a question of a generalization of the family constructed in [4] for higher dimensions is very natural. In this paper that question is positively resolved: for any real dimension $4(n+1)$ we construct in an explicit form the continuous family of metrics $\bar{G}_\alpha$ "connecting" Calabi metrics.

Theorem. *The following family consists of complete Ricci-flat $4(n+1)$-dimensional Riemannian metrics:*

$$\bar{G}_\alpha = \frac{r^4(r^4-\alpha^4)^n}{(r^4-\alpha^4)^{n+1}-(1-\alpha^4)^{n+1}}dr^2 + \frac{(r^4-\alpha^4)^{n+1}-(1-\alpha^4)^{n+1}}{r^2(r^4-\alpha^4)^n}\eta_1^2 + r^2(\eta_2^2+\eta_3^2)$$

$$+(r^2+\alpha^2)\sum_{\beta=1}^n(\eta_{4\beta}^2+\eta_{5\beta}^2) + (r^2-\alpha^2)\sum_{\beta=1}^n(\eta_{6\beta}^2+\eta_{7\beta}^2),$$

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where $0 \le \alpha \le 1$, $r \ge 1$. Metrics $\bar{G}_0$ and $\bar{G}_1$ have holonomies $SU(2(n+1))$ and $Sp(n+1)$ accordingly and coincide with high-dimensional Calabi metrics found in [5]. Metrics $\bar{G}_\alpha$ for $0 < \alpha < 1$ have holonomy $SU(2(n+1))$ and for $n = 1$ coincide with the family constructed in [4]. For $0 < \alpha < 1$ metrics $\bar{G}_\alpha$ are defined on the $(n+1)$th tensor power of the complex line bundle over the space of complex flags in $\mathbb{C}^{2n+1}$ and metric $\bar{G}_1$ is defined on $T^*\mathbb{C}P^{n+1}$.

In the next section we will explain in detail construction of the metrics $\bar{G}_\alpha$ and will prove the above theorem.

2 The Proof.

In paper [4] we explored the existence of the 8-dimensional metrics with holonomy $Spin(7)$ of the following form

$$dt^2 + A_1(t)^2\eta_1^2 + A_2(t)^2\eta_2^2 + A_3(t)^2\eta_3^2 + B(t)^2(\eta_4^2+\eta_5^2) + C(t)^2(\eta_6^2+\eta_7^2)$$

on the cone over 7-dimensional 3-Sasakian manifold M, whose 4-dimensional quaternionic Kähler orbifold $\mathcal{O}$ possesses a Kähler structure. The form dt corresponds to the generator of the cone, the forms $\eta_i,\ i = 1,2,3$ are the characteristic forms of the 3-Sasakian manifold and the forms $\eta_i,\ i = 4,\dots,7$ are 1-forms on the orbifold. The condition for holonomy to be contained in $Spin(7)$ is equivalent to some system of ODE's on the functions A_i, B, C. This system was explored carefully and we were able to find the following family of solutions:

$$\begin{aligned} \bar{g}_\alpha = {} & \tfrac{r^4(r^2-\alpha^2)(r^2+\alpha^2)}{r^8-2\alpha^4(r^4-1)-1}dr^2 + \tfrac{r^8-2\alpha^4(r^4-1)-1}{r^2(r^2-\alpha^2)(r^2+\alpha^2)}\eta_1^2 + r^2(\eta_2^2+\eta_3^2) \\ & +(r^2+\alpha^2)(\eta_4^2+\eta_5^2) + (r^2-\alpha^2)(\eta_6^2+\eta_7^2), \end{aligned} \qquad (*)$$

where $0 \le \alpha \le 1$ and $r \ge 1$. Metrics $(*)$ are defined on a smooth manifold if and only if M is the Aloff-Wallach space $N_{1,1} = SU(3)/S^1$.

Calabi constructed his metrics $\bar{G}_0$ and $\bar{G}_1$ on the complex bundles over the Kähler-Einstein manifolds [5]. In particular, metrics with holonomy $SU(n)$ were constructed on the line bundles over the compact Kähler-Einstein manifolds and the hyperkähler metrics were constructed on the $T^*\mathbb{C}P^n$. Nevertheless in [5] this metrics were not written out in the explicit form.

An expression for the metric $\bar{G}_0$ was found in [6]:

$$[1-(\frac{1}{\rho})^{2m+2}]^{-1}d\rho^2 + [1-(\frac{1}{\rho})^{2m+2}]\rho^2(d\tau-2A)^2 + \rho^2 ds^2, \qquad (1)$$

where ds^2 is a metric on a 3-dimensional Hodge and Kähler-Einstein manifold F, dA — Kähler form on F. For $m = 3$ and $F = SU(3)/T^2$ metrics (1) and $\bar{g}_0$ coincide. Metric (1) is defined on the $(m+1)$th power of the canonical line bundle over F.

In [7] it was attempted to explore the metrics of cohomogeneity one on the $T^*\mathbb{C}P^{(n+1)}$. It's not difficult to understand the spherical subbundle in $T^*\mathbb{C}P^{(n+1)}$ fibres over the space $SU(n+2)/(U(n)\times U(1))$. On the Lie algebra $su(n+1)$ one can choose the cobasis of the left-invariant 1-forms L_A^B such that its exterior algebra satisfies $dL_A^B = iL_A^C \wedge L_C^B$. Index A takes values in $(1, 2, \beta)$ and index β takes values from 1 to n, further β is never fixed and no ambiguity appears. Obviously, $u(n) \oplus u(1)$ is a Lie subalgebra in $su(n+2)$ and is not an exterior subalgebra. Forms $L_1^\beta = \sigma_\beta$, $L_2^\beta = \Sigma_\beta$ and $L_1^2 = \nu$ constitute a basis on the quotient $su(n+2)/(u(n)\oplus u(1))$. Then one can define the real forms: $\sigma_{1\beta} + i\sigma_{2\beta} = \sigma_\beta$, etc. Form $\lambda = L_1^1 - L_2^2$ is real by definition. In [7] the metrics of the following form

$$dt^2 + a(t)^2|\sigma_\beta|^2 + b(t)^2|\Sigma_\beta|^2 + c(t)^2|\nu|^2 + f(t)^2\lambda^2, \tag{2}$$

were considered, where the summation over the β from 1 to n is omitted. The expression for hyperkähler metric $\bar{G}_1$ on the $T^*\mathbb{C}P^{n+1}$ was found:

$$\frac{dr^2}{1-r^{-4}} + \frac{1-r^{-4}}{4}r^2\lambda^2 + r^2(\nu_1^2+\nu_2^2) + \frac{r^2+1}{2}(\Sigma_{1\beta}^2+\Sigma_{2\beta}^2) + \frac{r^2-1}{2}(\sigma_{1\beta}^2+\sigma_{2\beta}^2). \tag{3}$$

In the case $n = 1$ to make the notations of papers [4] and [7] agree one should put

$$\lambda = 2\eta_1,\ \nu_1 = \eta_3,\ \nu_2 = \eta_2,\ \Sigma_1 = \sqrt{2}\eta_4,\ \Sigma_2 = \sqrt{2}\eta_5,\ \sigma_1 = \sqrt{2}\eta_6,\ \sigma_2 = \sqrt{2}\eta_7.$$

In [7] also the Ricci tensor were written out. Obviously this tensor has five components: $Ric = R_0 dt^2 + R_a|\sigma_\beta|^2 + R_b|\Sigma_\beta|^2 + R_c|\nu|^2 + R_f\lambda^2$ and depends on four functions. We don't write it out here.

Notice that for any dimension n coefficients of metric (3) have the same form, and coefficients of (1) depend on n explicitly. Therefore, the requested family of metrics should depend on n explicitly and for $\alpha = 1$ should coincide with (3). We will look for metrics of the following form:

$$\frac{dr^2}{W^2} + \frac{W^2r^2}{4}\lambda^2 + r^2(\nu_1^2+\nu_2^2) + \frac{(r^2-\alpha^2)}{2}(\sigma_{1\beta}^2+\sigma_{2\beta}^2) + \frac{(r^2+\alpha^2)}{2}(\Sigma_{1\beta}^2+\Sigma_{2\beta}^2),$$

where $W = W(r, n, \alpha)$ is an unknown function. If one will put appropriate functions into the expressions for the Ricci tensor then the components

$(R_a,\ R_b,\ R_c)$ will be

$$R_a = -\frac{2Q}{(r^2-\alpha^2)^2(r^2+\alpha^2)}$$
$$R_b = \frac{2Q}{(r^2+\alpha^2)^2(r^2-\alpha^2)}$$
$$R_c = -\frac{2Q}{r^2(r^4-\alpha^4)},$$

where $Q = \frac{dW}{dr}(r^5 - r\alpha^4) + 4W^2\alpha^4 + 4(n+1)(r^4 - \alpha^4 - r^4W^2)$ and $\frac{dr}{dt} = W$. This equation can be integrated without difficulties:

$$W^2 = \frac{(r^4-\alpha^4)^{n+1} + C}{r^4(r^4-\alpha^4)^n},$$

where C is an integration constant. By a shift along r this constant can be fixed. One should put $C = -(1-\alpha^4)^{n+1}$ then $r \geq 1$ and $W(1) = 0$. The components R_0 and R_f are the second order ODE's and automatically vanish.

Consider the following 2-form

$$\Omega = rdr \wedge \lambda + 2r^2\nu_1 \wedge \nu_2 - (r^2+\alpha^2)\Sigma_{1\beta} \wedge \Sigma_{2\beta} + (r^2-\alpha^2)\sigma_{1\beta} \wedge \sigma_{2\beta}.$$

Using the exterior algebra's relations from [7] one can easily verify that this form is closed and up to multiplying by $\frac{1}{2}$ is the Kähler form of metric (2). The vanishing of the Ricci tensor and the closeness of the form Ω give us the statement about holonomy.

Next we will explore the topology of the spaces where the founded metrics are defined. Here we generalize construction for higher dimensions used in [4]. Consider the complex space $\mathbb{C}^{n+2}$ and the diagonal action of a circle S^1 on it. This action defines an equivalence class. We will designate such a class by square brackets. For example, $[u]$, $[V]$ where $u,\ V$ are vectors or subspaces.

Consider the space $\tilde{H} = \{(u_1, u_2, V)|\ |u_1| = 1, u_1 \perp_{\mathbb{C}} u_2 \perp_{\mathbb{C}} V\} \subset S^{2n+3} \times \mathbb{C}^{n+2} \times G_n(\mathbb{C}^{n+2})$. Consider also the projection $\tilde{\pi} : (u_1, u_2, V) \to (u_1, V)$ from $\tilde{H}$ to the space $\tilde{F} = \{(u_1, V)|\,|u_1| = 1, u_1 \perp_{\mathbb{C}} V\}$. The spherical subbundle $\tilde{H}^1 = \{(u_1, u_2, V)|\ |u_i| = 1, u_1 \perp_{\mathbb{C}} u_2 \perp_{\mathbb{C}} V\}$ can be identified with $SU(n+2)/SU(n)$. By using the diagonal action of the S^1 and $\tilde{\pi}$ one gets the complex line bundle $\pi : H = \tilde{H}/S^1 \to F = \tilde{F}/S^1$. The spherical subbundle in π coincides with the map $\pi^1 : H^1 = SU(n+2)/S[U(n) \times U(1)] \to F = SU(n+2)/T$, where

$$T = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & \bar{z}\det\bar{A} & 0 \\ 0 & 0 & A \end{pmatrix} |z \in U(1), A \in U(n) \right\}.$$

Notice that H^1 is a 3-Sasakian manifold and coincides with Aloff-Wallach space for $n = 1$ and π^1 is its fibration over the respective twistor space $\mathcal{Z} = F$ — the space of complex flags $\{([u], V)|\ u \in S^{2n+3}, V \in G_n(\mathbb{C}^{n+2}), u \perp_{\mathbb{C}} V\}$ in $\mathbb{C}^{n+2}$. It is not difficult to verify that the length with respect to the metric $\bar{G}_\alpha$ of the characteristic vector field dual to the form η_1 at the start time $r = 1$ is equal to $2(n + 1)$. For metric $\bar{G}_\alpha$ to be well-defined it is necessary for the circle generated by that vector field to be factorized by the discrete subgroup $\mathbb{Z}_{n+1}$ because in the 3-Sasakian fibration π^1 there is already factorization by $\mathbb{Z}_2$ (look [1] for details). Thus, the metrics $\bar{G}_\alpha$ for $0 \leq \alpha < 1$ are defined on the tensor power π^{n+1}. The theorem is proved.

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